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Comments on double scaled little string theory

Amit Giveon[ab] **and David Kutasov**[ac]

[a]*Institute for Theoretical Physics, University of California*
Santa Barbara, CA 93106, USA

[b]*Racah Institute of Physics, The Hebrew University*
Jerusalem 91904, Israel

[c]*Department of Physics, University of Chicago*
5640 S. Ellis Av., Chicago, IL 60637, USA

E-mail: giveon@vms.huji.ac.il, kutasov@theory.uchicago.edu

Abstract: We study little string theory in a weak coupling limit defined in [1].

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Contents

1. Introduction

Little String Theories (LST's) are non-local theories without gravity which can be defined by studying the dynamics on NS5-branes in the limit in which the string coupling $g_s \to 0$ [2] (see also [3, 4]). An alternative (and possibly more general) definition of these theories is obtained by studying string theory on

$$\mathbb{R}^{d-1,1} \times \mathcal{M}^{2n}\,, \tag{1.1}$$

where $\mathcal{M}^{2n}$ is a singular Calabi-Yau (CY) n-fold ($2n = 10 - d$). In the limit $g_s \to 0$ the modes that propagate in the bulk of $\mathcal{M}^{2n}$ decouple and one finds a non-trivial theory living at the singular locus of $\mathcal{M}^{2n}$. In some cases the two definitions are related by T-duality [5, 6, 7, 8].

A large class of examples is obtained by studying CY n-folds $\mathcal{M}^{2n}$ (1.1) with an isolated quasi-homogenous hypersurface singularity. Near such a singularity, $\mathcal{M}^{2n}$ can be described as the hypersurface $F(z_1, \ldots, z_{n+1}) = 0$ in $\mathbb{C}^{n+1}$, where F is a quasi-homogenous polynomial with weight one under $z_a \to \lambda^{r_a} z_a$, i.e.

$$F(\lambda^{r_a} z_a) = \lambda F(z_a)\,, \qquad \lambda \in \mathbb{C}\,, \tag{1.2}$$

for some set of positive weights r_a.

Finding a useful description of LST's is an important open problem. In [9] it was proposed to study them using holography (see also [10]–[17]). In [18] it was further argued that the non-gravitational d-dimensional theory at a general singularity $F(z_a) = 0$ is dual to string theory in the background

$$\mathbb{R}^{d-1,1} \times \mathbb{R}_\phi \times S^1 \times LG(W = F)\,, \tag{1.3}$$

where $\mathbb{R}_\phi$ is the real line, labeled by ϕ, along which the dilaton varies linearly,

$$\Phi = -\frac{Q}{2}\phi\,, \tag{1.4}$$

and $LG(W = F)$ is a Landau-Ginzburg $N = 2$ SCFT of $n + 1$ chiral superfields z_a with superpotential $W(z_a) = F(z_a)$ (1.2). *Off-shell* physical observables in LST correspond to *on-shell* observables in string theory on (1.3). Off-shell Green's functions in LST correspond to on-shell correlation functions in the dual string theory.

The description (1.3) is useful at large ϕ where the string coupling $e^\Phi \to 0$ and the theory is weakly coupled. One can use it to identify off-shell observables in LST (which correspond in the dual description to wavefunctions which are exponentially supported at $\phi \to \infty$) and study their transformation properties under the symmetries of the theory [9]. Correlation functions are governed by the region of finite ϕ, and therefore computing them requires knowledge of the strong coupling dynamics at $\phi \to -\infty$. Perturbative string theory in the vacuum (1.3) is singular and does not provide a good guide for such correlation functions.

The strong coupling singularity at $\phi \to -\infty$ in (1.3), (1.4) can be avoided by replacing the cylinder $\mathbb{R}_\phi \times S^1$ in (1.3) by the SCFT on the "cigar" SL(2)/ U(1) (or equivalently [1] $N = 2$ Liouville). In the dual theory this corresponds [18, 1] to resolving the singularity $F(z_a) = 0$ to

$$F(z_1, \ldots, z_{n+1}) = \mu\,, \tag{1.5}$$

and taking the limit $\mu \to 0$, $g_s \to 0$ with the ratio

$$\chi \equiv \frac{\mu^{r_\Omega}}{g_s} \tag{1.6}$$

held fixed. Here

$$r_\Omega \equiv \sum_{a=1}^{n+1} r_a - 1 = \frac{Q^2}{2} \tag{1.7}$$

(see [18, 1] for the details). The value of the string coupling at the tip of the cigar is $1/\chi$; hence, by making χ large one obtains a weakly coupled theory, Double Scaled Little String Theory (DSLST), in which correlation functions have a $1/\chi$ expansion.

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To summarize, the non-gravitational d-dimensional theory which lives at the resolved singularity (1.5) in the double scaling (decoupling) limit $\mu, g_s \to 0$ with χ (1.6) held fixed is dual to string theory in the background

$$\mathbb{R}^{d-1,1} \times \frac{\mathrm{SL}(2)_k}{\mathrm{U}(1)} \times LG(W = F)\,, \tag{1.8}$$

where k is the level of SL(2) (it is related to (1.7), $k = 1/r_\Omega$ [18]). One can think of (1.8) as the "near horizon geometry" of the resolved singularity $F = \mu$ (1.5) [19].

An interesting example of the construction is the $d = 6$ theory of k parallel NS5-branes. In that case one has

$$F(z_1, z_2, z_3) = z_1^k + z_2^2 + z_3^2\,, \tag{1.9}$$

and the manifold $F(z_a) = 0$ is an ALE space (the singular geometry which is T-dual to the fivebrane near horizon background). The low energy limit of the theory on k NS5-branes in type-IIB string theory is SU(k) gauge theory with $(1,1)$ SUSY (a non-chiral theory with sixteen supercharges); for type-IIA fivebranes one finds at low energies a mysterious non-abelian generalization of the theory of self-dual $B_{\mu\nu}$ fields, with chiral $(2,0)$ SUSY and conformal invariance (see e.g. [20] for a review). At high energies both the IIA and IIB theories are expected to have a Hagedorn density of states, but unlike ordinary critical string theories they do not contain gravity and do have well defined off-shell Green's functions.

The dual background (1.3) is in this case $\mathbb{R}^{5,1} \times \mathbb{R}_\phi \times S^1 \times \mathrm{SU}(2)_k/\,\mathrm{U}(1)$. The GSO projection acts as a $\mathbb{Z}_k$ orbifold on $S^1 \times \mathrm{SU}(2)_k/\,\mathrm{U}(1)$, turning it into $\mathrm{SU}(2)_k$ WZW, the CFT on a three-sphere. This theory has an $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry which in the fivebrane language is the SO(4) R-symmetry of rotations in the $\mathbb{R}^4$ transverse to the branes [21].

Turning on μ (1.5) corresponds in the fivebrane language to distributing the fivebranes at equal distances around a circle of radius $r_0 \sim l_s \mu^{1/k}$ (see [1] for details). Parametrizing the $\mathbb{R}^4$ transverse to the branes by two complex variables A, B, the fivebrane theory includes two $k \times k$ matrices of complex scalar fields which we also denote by A, B. These scalars belong to the same SUSY multiplets as the SU(k) gauge field in type IIB, and the non-abelian self-dual $B_{\mu\nu}$ field and an additional real adjoint scalar in the IIA theory.

The double scaling limit (1.5)–(1.7) corresponds to studying the theory at a point in its Coulomb branch where, say,

$$\begin{aligned}\langle A \rangle &= 0 \\ \langle B \rangle &= M_W M_s \,\mathrm{diag}(e^{2\pi i/k}, e^{4\pi i/k}, \ldots, e^{2\pi i k/k})\,.\end{aligned} \tag{1.10}$$

In type-IIB string theory, this Higgses the SU(k) gauge symmetry down to $\mathrm{U}(1)^{k-1}$ with the mass of the charged gauge bosons $M_{jl} \sim M_W |e^{2\pi i j/k} - e^{2\pi i l/k}|$.

These gauge bosons correspond to the ground states of D-strings stretched between the NS5-branes; thus, M_W is related to r_0 via

$$M_W \sim \frac{r_0}{g_s l_s^2} \sim \chi M_s \,. \tag{1.11}$$

Therefore, the double scaling limit is in this case the decoupling limit with the mass of the W-bosons held fixed.

In the IIA theory we have instead $D2$-branes stretched between the NS5-branes; these look like strings in the fivebranes, charged under the respective self-dual $B_{\mu\nu}$ fields on the fivebranes. The tensions of these strings are proportional to $M_{jl}M_s$. They are held finite in the double scaling limit.

Distributing the branes as in (1.10) breaks the R-symmetry

$$\mathrm{SO}(4) \to \mathrm{SO}(2) \times \mathbb{Z}_k \,. \tag{1.12}$$

The charges of A and B under these symmetries are $(1,0)$ and $(0,1)$, respectively. In the dual picture (1.8), the SO(2) and $\mathbb{Z}_k$ charges correspond to momentum and winding around the cigar, respectively[1] (recall that in the cigar CFT, momentum around the cigar is conserved, while winding number is not).

2. Observables in DSLST

We are interested in constructing BRST invariant observables in string theory on (1.8). We will focus on (NS,NS) sector fields, since the rest of the observables can be obtained from these by applying the spacetime supercharges [18]. Such observables are linear combinations of vertex operators of the form (see [1, 18] for details)

$$O_{W,m,\bar{m}}(k_\mu) = e^{-\varphi-\bar{\varphi}} W_{\Delta,\bar{\Delta}} e^{ik_\mu x^\mu} V_{j;m,\bar{m}} \,, \tag{2.1}$$

where k_μ $(\mu = 0, 1, \ldots, d-1)$ is the momentum along $\mathbb{R}^{d-1,1}$ and $W_{\Delta,\bar{\Delta}}$ is a rather general operator[2] with scaling dimension $(\Delta, \bar{\Delta})$ in the SCFT on (1.8). $V_{j;m,\bar{m}}$ is a Virasoro primary on $\mathrm{SL}(2)/\mathrm{U}(1)$. Its scaling dimensions are

$$\Delta_{j;m,\bar{m}} = \frac{m^2 - j(j+1)}{k}\,; \qquad \bar{\Delta}_{j;m,\bar{m}} = \frac{\bar{m}^2 - j(j+1)}{k}\,, \tag{2.2}$$

where $(m, \bar{m})$ run over the set

$$m = \frac{1}{2}(p + wk)\,; \qquad \bar{m} = -\frac{1}{2}(p - wk)\,. \tag{2.3}$$

[1] In the language of [18], the SO(2) and $\mathbb{Z}_k$ charges are $\frac{1}{2}(R-\bar{R})$ and $\frac{1}{2}(R+\bar{R})$ mod k, respectively.

[2] By a gauge choice it can be set to be the identity on the cigar, and it will be convenient to do that below.

One can think of p as the momentum number around the cigar (which is conserved) and w as the winding number (which is not). In CFT on $\mathrm{SL}(2)/\mathrm{U}(1)$ both p and w are integers. String theory on (1.8) further includes a chiral orbifold (the GSO projection) acting non-trivially on $\mathrm{SL}(2)_k/\mathrm{U}(1)\times LG(W=F)$ in (1.8). The twisted sectors contain in general operators with non-integer winding number w which is conserved modulo one; p remains integer.

The on-shell condition requires (the coefficient of $e^{-\varphi-\bar{\varphi}}$ in) (2.1) to be a bottom component of a worldsheet $N=1$ superfield, and in addition to satisfy (we will usually set $M_s=1$ from now on)

$$\begin{aligned}\frac{1}{2}k_\mu k^\mu+\frac{m^2-j(j+1)}{k}+\Delta&=\frac{1}{2}\,,\\ \frac{1}{2}k_\mu k^\mu+\frac{\bar{m}^2-j(j+1)}{k}+\bar{\Delta}&=\frac{1}{2}\,.\end{aligned}\tag{2.4}$$

The chiral GSO projection further requires

$$Q_{LG}+\frac{2m}{k}+F_L\in 2\mathbb{Z}+1\,,\tag{2.5}$$

where Q_{LG} is the charge of $W_{\Delta,\bar{\Delta}}$ under the $\mathrm{U}(1)_R$ symmetry which belongs to the left-moving $N=2$ superconformal algebra in $LG(W=F)$, and F_L is the left-moving fermion number on $\mathbb{R}^{d-1,1}$. A similar relation holds for the right-movers. Unitarity of the "near horizon CFT," which is an orbifold of $\mathrm{SL}(2)/\mathrm{U}(1)\times LG(W=F)$, implies that j belongs to one of the following two regions (for early work on unitarity of $\mathrm{SL}(2)/\mathrm{U}(1)$ see [22]):

$$\begin{aligned}&j\in\mathbb{R}:\quad -\frac{1}{2}<j<\frac{k-1}{2}\,,\\ &\frac{m^2-j(j+1)}{k}+\Delta_{LG}\geq 0\,,\\ &\frac{\bar{m}^2-j(j+1)}{k}+\bar{\Delta}_{LG}\geq 0\,,\end{aligned}\tag{2.6}$$

or

$$j\in-\frac{1}{2}+i\mathbb{R}\,,\tag{2.7}$$

where Δ_{LG} is the contribution to Δ of the LG model in (1.8). Off-shell observables in the d-dimensional LST correspond to wave-functions on the cigar that are exponentially supported far from the tip, at $\phi\to\infty$. Therefore, only vertex operators in the range (2.6) give rise to such observables. Operators in the range (2.7) are δ function normalizable, and correspond to fluctuating fields in LST.

Note that the bound on j (2.6) implies via (2.4) that off-shell physical observables (2.1) are only defined in a finite range of values of $k_\mu k^\mu$. This issue will be further discussed below.

It is instructive to apply the above discussion to the six dimensional theory of k NS5-branes in type-II string theory. The string vacuum

$$\mathbb{R}^{5,1}\times\frac{\mathrm{SL}(2)_k}{\mathrm{U}(1)}\times\frac{\mathrm{SU}(2)_k}{\mathrm{U}(1)} \tag{2.8}$$

has the same R-symmetry, $\mathrm{SO}(2)\times\mathbb{Z}_k$, as the brane theory (1.12). The SO(2) charge is $m-\bar{m}=p$ (2.3), while the $\mathbb{Z}_k$ charge is $(m+\bar{m}) \bmod k$.

In the fivebrane theory there are two sets of chiral operators:[3]

$$\operatorname{Tr}A^i\,,\qquad \operatorname{Tr}B^i\,,\qquad i=2,\ldots,k\,. \tag{2.9}$$

The $\mathrm{SO}(2)\times\mathbb{Z}_k$ charge of A^i is $(i,0)$; that of B^i is $(0,i)$. In the string vacuum (2.8) they are realized as follows:

$$\operatorname{Tr}B^{2m}\leftrightarrow e^{-\varphi-\bar{\varphi}}z_1^{k-2m}e^{ik_\mu x^\mu}V_{j;m,m} \tag{2.10}$$

$$\operatorname{Tr}A^{2m}\leftrightarrow e^{-\varphi-\bar{\varphi}}\tilde{z}_1^{k-2m}e^{ik_\mu x^\mu}V_{j;m,-m} \tag{2.11}$$

$$2m=2,3,\ldots,k\,,\qquad\qquad k_\mu^2=\frac{2}{k}(j-m+1)(j+m)\,,$$

where z_1^i are the chiral-chiral operators in the $N=2$ minimal model $\mathrm{SU}(2)/\mathrm{U}(1)$ (see (1.9)), and $\tilde{z}_1^i$ are their chiral-antichiral analogs, which appear in the twisted sectors of the GSO orbifold. The following comments regarding the correspondence (2.10), (2.11) are in order at this point:

1. The $\mathrm{SO}(2)\times\mathbb{Z}_k$ (see (1.12)) charges of the vertex operators on the r.h.s. of (2.10) and (2.11) are $(0,2m)$ and $(2m,0)$, respectively, in agreement with those of $\operatorname{Tr}B^{2m}$ and $\operatorname{Tr}A^{2m}$.

2. The supersymmetry multiplet structure is the same. In particular, the vertex operators in (2.10) and (2.11) preserve the same halves of the SUSY as the operators on the l.h.s. [18].

3. Far from the tip of the cigar, where it looks like a cylinder, one recovers the background $\mathbb{R}_\phi\times S^3$ studied in [9]. Roughly speaking, large ϕ corresponds to high energies on the fivebrane, a regime in which the Higgs breaking due to (1.10) is negligible. The vertex operators[4] in (2.10), (2.11) approach for large ϕ the "graviton" operators given in [9, eq. (3.6)]. The identification (2.10), (2.11) agrees with the one proposed in [9] in that limit.

4. There are analogs of the vertex operators in (2.10) and (2.11) with $m\neq\bar{m}$. These correspond to $\operatorname{Tr}A^{m-\bar{m}}B^{m+\bar{m}}$. Such operators do not preserve any spacetime SUSY.

[3]The $k\times k$ matrix scalar fields A and B were defined before equation (1.10). In DSLST the off-diagonal components of A, B are massive; the eigenvalues of the matrices are the light degrees of freedom.

[4]Which can be thought of as "tachyons" from the d-dimensional perspective [18, 1].

3. The two point functions

The CFT on the cigar is related by the GKO coset construction [23] to that on AdS_3. The natural observables in CFT on the euclidean version of AdS_3 (the manifold $H_3^+ = \mathrm{SL}(2,\mathbb{C})/\,\mathrm{SU}(2)$) are functions $\Phi_j(x,\bar{x};z,\bar{z})$ which transform as primaries under the $\mathrm{SL}(2)_L \times \mathrm{SL}(2)_R$ current algebra [24, 25]. x is an auxiliary complex variable that is useful for studying CFT on AdS_3. The two point function of Φ_j is[5]

$$\langle \Phi_j(x,\bar{x})\Phi_j(x',\bar{x}')\rangle = \frac{k}{\pi}[\nu(k)]^{2j+1}\frac{\Gamma(1-\frac{2j+1}{k})}{\Gamma(\frac{2j+1}{k})}|x-x'|^{-4(j+1)}\,, \tag{3.1}$$

where

$$\nu(k) \equiv \frac{1}{\pi}\frac{\Gamma(1+1/k)}{\Gamma(1-1/k)}\,. \tag{3.2}$$

For studying the coset it is convenient to "Fourier transform" the fields $\Phi_j(x,\bar{x})$ and define the mode operators

$$\Phi_{j;m,\bar{m}} = \int d^2x\, x^{j+m}\bar{x}^{j+\bar{m}}\Phi_j(x,\bar{x})\,. \tag{3.3}$$

Note that for (3.3) to make sense one needs $m-\bar{m}\in\mathbb{Z}$; as discussed above (see (2.3)) this is indeed always the case.

The two point functions of the modes of Φ are equal to those of the $\mathrm{SL}(2)/\,\mathrm{U}(1)$ coset theory,

$$\langle V_{j;m,\bar{m}}V_{j;-m,-\bar{m}}\rangle = \langle \Phi_{j;m,\bar{m}}\Phi_{j;-m,-\bar{m}}\rangle \tag{3.4}$$

(the two differ by the two point function of exponentials in CFT on S^1, which is equal to one).[6]

The two point function of (2.1) is a generalization of the one computed in [1] for $m=\bar{m}$ using the results of [25]:

$$G_2(k_\mu) \equiv \langle O_{W,m,\bar{m}}(k_\mu)O_{W',-m,-\bar{m}}(-k_\mu)\rangle = \langle W_{\Delta,\bar{\Delta}}W'_{\Delta,\bar{\Delta}}\rangle\langle V_{j;m,\bar{m}}V_{j;-m,-\bar{m}}\rangle\,. \tag{3.5}$$

Using (3.4), (3.3), (3.1) and the results of appendix A one finds

$$\langle V_{j;m,\bar{m}}V_{j;-m,-\bar{m}}\rangle = k[\nu(k)]^{2j+1}\frac{\Gamma(1-\frac{2j+1}{k})\Gamma(-2j-1)\Gamma(j-m+1)\Gamma(1+j+\bar{m})}{\Gamma(\frac{2j+1}{k})\Gamma(2j+2)\Gamma(-j-m)\Gamma(\bar{m}-j)}\,. \tag{3.6}$$

The two point function (3.6) has a series of poles; these can be interpreted as contributions of on-shell states in DSLST, which are created from the vacuum by the

[5] Here and below we suppress the dependence of correlation functions on the worldsheet locations z, $\bar{z}$.

[6] $V_{j;m,\bar{m}}$ defined by (3.3), (3.4) is normalized such that far from the tip of the cigar it describes an incoming wave with strength one (see [1, eq. (3.10)]).

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operator $O_{W,m,\bar{m}}$. The masses of these states can be computed by using the relation (2.4) between j and $M^2 = -k_\mu k^\mu$. The resulting spectrum of masses can be analyzed as in [1]. Restricting to the region (2.6) gives rise to non-negative values of M^2 corresponding to

$$m = j + n; \quad n = 1, 2, 3, \ldots \tag{3.7}$$

or

$$\bar{m} = -(j + n); \quad n = 1, 2, 3, \ldots \tag{3.8}$$

These values of m and j belong to the principal discrete series representations of SL(2). The corresponding states can be thought of as bound states that live near the tip of the cigar [26]. Without loss of generality, we will restrict to the set (3.7).

Near $m = j + n$ the two point function (3.5) takes the form

$$G_2 = \frac{\langle W_{\Delta,\bar{\Delta}} W'_{\Delta,\bar{\Delta}} \rangle}{j + n - m} k[\nu(k)]^{2j+1} \frac{\Gamma(1 - \frac{2j+1}{k})\Gamma(1 + j + \bar{m})}{\Gamma(\frac{2j+1}{k})\Gamma(2j+2)\Gamma(\bar{m} - j)} \frac{1}{(n-1)!} \prod_{i=2}^{n} (2j + i) + \cdots , \tag{3.9}$$

where the "$\cdots$" correspond to terms analytic near $m = j + n$. Using (2.4), one can rewrite (3.9) as

$$G_2(k_\mu) = \frac{R_{n,m}}{k_\mu^2 + M_{n,m}^2} + \cdots , \tag{3.10}$$

where [1]

$$\frac{1}{k_\mu^2 + M_{n,m}^2} = \frac{k}{2(2j+1)(j+n-m)} , \tag{3.11}$$

$$M_{n,m}^2 = 2\left(\Delta + \frac{m}{k}\right) - 1 + \frac{2}{k}(n-1)(2m-n) ,$$
$$\text{with } 2m + 1 > 2n > 2m - k + 1 , \tag{3.12}$$

$$R_{n,m} = 2\langle W_{\Delta,\bar{\Delta}} W'_{\Delta,\bar{\Delta}} \rangle [\nu(k)]^{2j+1} \frac{\Gamma(1 - \frac{2j+1}{k})\Gamma(1 + j + \bar{m})}{\Gamma(\frac{2j+1}{k})\Gamma(2j+2)\Gamma(\bar{m} - j)} \frac{1}{(n-1)!} \prod_{i=1}^{n} (2j + i) . \tag{3.13}$$

It is useful to note that:

1. Since $m - \bar{m} = p$ (2.3), we have $\bar{m} = j + n - p$ (recall that $n, p \in \mathbb{Z}$).

2. When $|\bar{m}| \leq j$, the residue (3.13) vanishes due to a pole of $\Gamma(\bar{m} - j)$. This is in agreement with the fact that, as explained in [1], physical particles correspond to the principal discrete series representations (for which $|\bar{m}| \geq j + 1$).

3. If $n > p$, i.e. $\bar{m} = j + \bar{n}$, $\bar{n} = 1, 2, \ldots$, the residue (3.13) is positive for all $\bar{n}$, in agreement with the expected unitarity of the DSLST.

4. For $n \leq p$, comment (2) above implies that non-zero residues correspond to $p - n > 2j$ and, furthermore, one must have $2j \in \mathbb{Z}$, in which case the pole of $\Gamma(\bar{m} - j)$ in the denominator is cancelled by a pole of $\Gamma(1 + j + \bar{m})$ in the numerator. The sign of the residue (3.13) does not vary with n. It does depend on the observable under consideration as $(-)^{2m+1}$, however, this dependence is insignificant. For example, one can absorb it into the rules for hermitian conjugation:[7]

$$V^{\dagger}_{j;m,\bar{m}} = \begin{cases} V_{j;-m,-\bar{m}} & m\bar{m} > 0 \\ (-)^{2m+1} V_{j;-m,-\bar{m}} & m\bar{m} < 0 \end{cases} . \tag{3.14}$$

To summarize, the two point function (3.5) exhibits a series of single poles, all of whose residues have the same sign (for a given observable). This is consistent with the expected unitarity of the d-dimensional theory.

In addition to the series of poles described above, the two point function (3.5) has a branch cut in momentum space, since j is determined in terms of $k_\mu k^\mu$ by solving the quadratic equation (2.4). This branch cut starts at $j = -1/2$ and is due to the fact that the operators $O_{W,m,\bar{m}}$ can create from the vacuum the continuum of normalizable states in the range (2.7). However, as discussed above, non-fluctuating physical observables correspond only to operators in the range (2.6) and, therefore, the role of both the branch cut and the observables (2.7) is not clear.

The analysis of the mass spectrum provides another check of the identification of the chiral operators ([17]) in the six dimensional fivebrane theory, at the point (1.10) in its Coulomb branch, with the vertex operators (2.10) and (2.11). The two point functions of these operators have poles given by (3.12) with $\Delta(z_1^{k-2m}) = \Delta(\bar{z}_1^{k-2m}) = (k-2m)/2k$. The lowest lying states created by these operators from the vacuum correspond to $n = 1$ in (3.12), and are massless. Thus, the operators (2.10) and (2.11) create from the vacuum $4(k-1)$ real scalar fields with the quantum numbers of the massless scalars corresponding to ([17]) in the fivebrane theory.

There is however a puzzle associated with the above discussion. At low energies, the eigenvalues of the matrices A, B are expected to be free massless fields. In free field theory, operators like ([17]) do not create single particle states from the vacuum: $\mathrm{Tr}\, A^i$ (say) creates an i-particle state. Put differently, in free field theory the two point function $\langle \mathrm{Tr}\, A^i(k_\mu)\, \mathrm{Tr}\, A^i(-k_\mu)\rangle$ behaves at small momenta like $(k_\mu k^\mu)^{2i-3} \log k_\mu k^\mu$ and *not* $1/k_\mu^2$ as found in (3.5)–(3.13). We will discuss this puzzle further in section 5.

[7] A similar phenomenon occurs in SU(2) WZW, where the $SU(2)_L \times SU(2)_R$ symmetry implies that the two point functions satisfy the relation $\langle V_{j;m,\bar{m}} V_{j;-m,-\bar{m}}\rangle = (-)^{2m} \langle V_{j;m,-\bar{m}} V_{j;-m,\bar{m}}\rangle$. Unitarity of the CFT then implies that one should use conjugation rules that are similar to the ones that appear here. Another potential source of minus signs is the relation between observables on SL(2) and SL(2)/ U(1), (3.4).

4. The three point functions

To compute the couplings among the particle states with masses given by (3.12) we next turn to the three point functions of the off-shell observables (2.1),

$$G_3 \equiv \langle O_{W_3,m_3,\bar{m}_3} O_{W_2,m_2,\bar{m}_2} O_{W_1,m_1,\bar{m}_1} \rangle \, . \tag{4.1}$$

The momentum around the cigar, $m - \bar{m}$, is conserved; hence three point functions satisfy

$$m_1 + m_2 + m_3 = \bar{m}_1 + \bar{m}_2 + \bar{m}_3 \, . \tag{4.2}$$

As is standard in fermionic string theory, to compute such three point functions we take two of the operators to be in the -1 picture, and the third in the 0 picture. After evaluating the ghost contributions, one finds:

$$G_3 = \left\langle W_{\Delta_3,\bar{\Delta}_3} e^{ik_3\cdot x} V_{j_3;m_3,\bar{m}_3} W_{\Delta_2,\bar{\Delta}_2} e^{ik_2\cdot x} V_{j_2;m_2,\bar{m}_2} G_{-1/2} \bar{G}_{-1/2} W_{\Delta_1,\bar{\Delta}_1} e^{ik_1\cdot x} V_{j_1;m_1,\bar{m}_1} \right\rangle . \tag{4.3}$$

$G_{-\frac{1}{2}}$ and $\bar{G}_{-\frac{1}{2}}$ are the left and right-moving global $N = 1$ superconformal generators.

There are apriori four contributions to (4.3): each of $G_{-1/2}$ and $\bar{G}_{-\frac{1}{2}}$ can act either on $W_1 e^{ik_1\cdot x}$ or on $V_{j_1;m_1,\bar{m}_1}$. The "mixed terms" in which the two supercharges act in different sectors vanish due to (4.2) and the left and right-moving $\mathrm{U}(1)_R$ $N = 2$ superconformal symmetries. This leaves two contributions:

$$\begin{aligned} G_3 = &\langle W_{\Delta_3,\bar{\Delta}_3} e^{ik_3\cdot x} W_{\Delta_2,\bar{\Delta}_2} e^{ik_2\cdot x} G_{-1/2} \bar{G}_{-1/2} W_{\Delta_1,\bar{\Delta}_1} e^{ik_1\cdot x} \rangle \langle V_{j_3;m_3,\bar{m}_3} V_{j_2;m_2,\bar{m}_2} V_{j_1;m_1,\bar{m}_1} \rangle + \\ &+ \langle W_{\Delta_3,\bar{\Delta}_3} e^{ik_3\cdot x} W_{\Delta_2,\bar{\Delta}_2} e^{ik_2\cdot x} W_{\Delta_1,\bar{\Delta}_1} e^{ik_1\cdot x} \rangle \langle V_{j_3;m_3,\bar{m}_3} V_{j_2;m_2,\bar{m}_2} G_{-1/2} \bar{G}_{-1/2} V_{j_1;m_1,\bar{m}_1} \rangle . \end{aligned} \tag{4.4}$$

The $\mathrm{U}(1)_R$ symmetries imply that the first term vanishes unless $m_1 + m_2 + m_3 = 0$. Thus, it corresponds to amplitudes for which both momentum (4.2) and winding are conserved.

The second term in (4.4) is in general non-zero when $m_1 + m_2 + m_3 = \pm k/2$. The $N = 1$ superconformal generator $G_{-1/2}$ can be decomposed into eigenstates of $\mathrm{U}(1)_R$, $G_{-1/2} = G^+_{-1/2} + G^-_{-1/2}$. The $\mathrm{U}(1)_R$ constraints imply that in the second term in (4.4) one has to act on V either with $G^+_{-1/2}\bar{G}^+_{-1/2}$ or with $G^-_{-1/2}\bar{G}^-_{-1/2}$. The resulting amplitudes violate winding number by one unit. It is interesting to note that:

1. One can generalize the above arguments to n point functions. The correlation function receives contributions that violate winding number by i units, with $i = 0, 1, \ldots, n-2$.

2. A related result was obtained by V. Fateev, A.B. Zamolodchikov and Al.B. Zamolodchikov (unpublished), who showed that the bosonic CFT on $\mathrm{SL}(2)/\mathrm{U}(1)$ has similar properties: n point functions can violate winding number conservation by up to $n - 2$ units.

We next turn to the calculation of the three point function (4.4). For simplicity, we will present the result for the case where all three momenta $m_i - \bar{m}_i$ vanish, and the winding number is conserved. In that case, only the first line of (4.4) contributes. Furthermore, the analytic structure is determined by the three point function on the cigar,

$$\langle V_{j_3;m_3,m_3} V_{j_2;m_2,m_2} V_{j_1;m_1,m_1} \rangle \,, \tag{4.5}$$

and we will focus on it below.

In the interpretation of the correlation function G_3 as an off-shell Green's function in DSLST, one expects (4.4), (4.5) to have poles at values of m and j corresponding to the principal discrete series (3.7), (3.8). Such poles would be due to the states found in section 3 (see (3.12)) going on mass shell. As in local QFT, LSZ reduction relates the residue of these poles to the S-matrix element describing (say) the decay of one of these particles to two others. Below we will compute these matrix elements.

The three point function (4.5) is obtained from the appropriate correlation function of $\mathrm{SL}(2)_L \times \mathrm{SL}(2)_R$ primaries in CFT on H_3^+, using the transform (3.3). As shown in [25],

$$\begin{aligned} &\langle \Phi_{j_3}(x_3,\bar{x}_3)\Phi_{j_2}(x_2,\bar{x}_2)\Phi_{j_1}(x_1,\bar{x}_1)\rangle = \\ &\quad = D(j_3,j_2,j_1;k)|x_1-x_2|^{2(j_3-j_1-j_2-1)}|x_1-x_3|^{2(j_2-j_1-j_3-1)}|x_2-x_3|^{2(j_1-j_2-j_3-1)} \,, \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} D(j_3,j_2,j_1;k) &= \frac{k}{2\pi^3}\nu(k)^{j_1+j_2+j_3+1} \times \\ &\times \frac{G(-j_1-j_2-j_3-2)G(j_3-j_1-j_2-1)G(j_2-j_1-j_3-1)G(j_1-j_2-j_3-1)}{G(-1)G(-2j_1-1)G(-2j_2-1)G(-2j_3-1)} \,. \end{aligned} \tag{4.7}$$

$G(j)$ is a special function, whose only properties that will be needed here are (see also appendix A):

1. It satisfies the recursion relation
$$G(j-1) = \frac{\Gamma(1+j/k)}{\Gamma(-j/k)} G(j) \,. \tag{4.8}$$

2. $G(j)$ has poles at the following values of j: $j = n+mk$, $j = -(n+1)-(m+1)k$, $n, m = 0, 1, 2, \ldots$

A potentially alarming aspect of (4.7) and (4.8) is the appearance of singularities in the three point couplings in string theory on AdS_3. One expects the spacetime theory to be a unitary CFT in which such singularities are unacceptable. Upon a closer look one actually finds that some singularities are acceptable. Consider, for example, the factor $G(j_1-j_2-j_3-1)$ in the numerator of (4.8). It has poles when $j_1-j_2-j_3-1 = n+mk$ or $-(n+1)-(m+1)k$ with $n, m = 0, 1, 2, \ldots$ The bound (2.6) satisfied by j_1, j_2 and j_3 eliminates most of these poles and allows only those with

$$j_1 - j_2 - j_3 - 1 = n = 0, 1, 2, \dots \tag{4.9}$$

These poles are harmless and even to some extent necessary. Note that when (4.9) is valid, the power of $|x_2-x_3|^2$ in (4.7) is a non-negative integer, n. Unless the structure constant $D(j_1, j_2, j_3; k)$ develops a pole in this case, the correlation function (4.7) is supported at zero momentum, since it is the Fourier transform of a polynomial in $x_2 - x_3$. The pole at (4.9) replaces

$$\frac{1}{\epsilon}|x_2 - x_3|^{2(n+\epsilon)} \to |x_2 - x_3|^{2n} \log |x_2 - x_3|^2 \,, \tag{4.10}$$

whose Fourier transform is a power of momentum, as required. The poles (4.9) will play an important role below.

Poles of (4.8) that cannot be interpreted as above (i.e. poles which persist after (4.7) is Fourier transformed to momentum space) are genuine pathologies, and must be avoided. An example of such poles appeared already in the two point function (3.1) for $j \geq (k-1)/2$; those poles were excluded by the unitarity bound (2.6) (in fact, excluding these poles is one way of deriving the bound [1]).

The three point function (4.7) has such poles when $j_1 + j_2 + j_3 = n + k - 1$, $n = 0, 1, 2, \dots$ These poles have to be excluded by unitarity as well. Thus, we conclude that unitarity of weakly coupled string theory on AdS_3 allows one to consider only three point functions (4.7) with[8]

$$j_1 + j_2 + j_3 < k - 1 \,. \tag{4.11}$$

We next use the transform (3.3) and the fact that the three point function satisfies

$$\langle V_{j_3;m_3,\bar m_3} V_{j_2;m_2,\bar m_2} V_{j_1;m_1,\bar m_1} \rangle = \langle \Phi_{j_3;m_3,\bar m_3} \Phi_{j_2;m_2,\bar m_2} \Phi_{j_1;m_1,\bar m_1} \rangle \,, \tag{4.12}$$

like the two point function[9] (3.4). We find

$$\begin{aligned} \langle V_{j_3;m_3,m_3} V_{j_2;m_2,m_2} V_{j_1;m_1,m_1} \rangle \;=\; & D(j_3, j_2, j_1; k) \times \\ & \times F(j_3, m_3; j_2, m_2; j_1, m_1) \int d^2x |x|^{2(m_1+m_2+m_3-1)} \,, \end{aligned} \tag{4.13}$$

where

$$\begin{aligned} F(j_3, m_3; j_2, m_2; j_1, m_1) \;=\; & \int d^2x_1\, d^2x_2 |x_1|^{2(j_1+m_1)} |x_2|^{2(j_2+m_2)} |1 - x_1|^{2(j_2-j_1-j_3-1)} \times \\ & \times |1 - x_2|^{2(j_1-j_2-j_3-1)} |x_1 - x_2|^{2(j_3-j_1-j_2-1)} \,. \end{aligned} \tag{4.14}$$

The integral over x in (4.14) ensures winding number conservation $m_1 + m_2 + m_3 = 0$. The function F (4.14) does not seem to be expressible in terms of elementary functions.

[8] This constraint on the observables in three point functions may seem puzzling. We will return to it in the next section.

[9] The situation for $n \geq 4$ point is more complicated.

To study the decay of on-shell states we must perform LSZ reduction of the three point function, i.e. evaluate F in the vicinity of its singularities. Specifically, since we are interested (say) in the decay process $1 \to 2+3$, we focus on the residue of the first order pole at

$$m_1 = -(j_1+n_1)\,, \qquad n_1 = 1,2,3,\ldots \tag{4.15}$$

which arises from the integration over x_1 near zero in (4.14). This corresponds to an incoming particle with mass determined by (3.12) (compare to (3.8)). One can show that the residue of the pole at (4.15) has further singularities in the other external legs. In particular, one finds single poles in j_2, j_3 at

$$m_2 = j_2+n_2\ ; \quad m_3 = j_3+n_3\,, \qquad n_2,n_3 = 1,2,3,\ldots \tag{4.16}$$

corresponding to the outgoing particles 2 and 3. Thus, near (4.15) and (4.16), the correlator (4.12) has the structure

$$\langle V_{j_3;m_3,m_3}V_{j_2;m_2,m_2}V_{j_1;m_1,m_1}\rangle = \frac{R_{3,2,1}}{(j_1+n_1+m_1)(j_2+n_2-m_2)(j_3+n_3-m_3)}+\cdots \tag{4.17}$$

We will next compute the residue $R_{3,2,1}$. Winding number conservation, (4.15) and (4.16) imply that

$$j_1-j_2-j_3 = n_2+n_3-n_1 \equiv N+1 \tag{4.18}$$

is an integer, which we denote by $N+1$.

As described in appendix B, the residue $R_{3,2,1}$ (4.17) vanishes for $N<0$. Thus, the decay amplitude $1\to 2+3$ vanishes in this case. For $N\geq 0$ one finds (using results from the appendices):

$$\begin{aligned} R_{3,2,1} = {} & \frac{k^2}{2\pi}\nu(k)^{j_1+j_2+j_3+1}S(2;3)S(3;2)\times \\ & \times\prod_{i=1}^{N}\frac{\Gamma(-\frac{i}{k})\Gamma(\frac{2j_1+1-i}{k})}{\Gamma(1+\frac{i}{k})\Gamma(1-\frac{2j_1+1-i}{k})}\prod_{i=0}^{N}\frac{\Gamma(1-\frac{2j_2+1+i}{k})\Gamma(1-\frac{2j_3+1+i}{k})}{\Gamma(\frac{2j_2+1+i}{k})\Gamma(\frac{2j_3+1+i}{k})}\,, \end{aligned} \tag{4.19}$$

where

$$\begin{aligned} S(2;3) = {} & \sum_{n=\max\{0,N+1-n_2\}}^{\min\{N,n_3-1\}}\binom{N}{n}\frac{(-)^{n_3-1-n}}{(n_3-1-n)!(n_2-1+n-N)!}\times \\ & \times\prod_{i=0}^{n_3-n-2}(2j_3+n_3+N-n-i)!\prod_{i=0}^{n_2+n-N-2}(2j_2+n_2+n-i)! \end{aligned} \tag{4.20}$$

$R_{3,2,1}$ (4.17), (4.19) times the contribution of $\mathbb{R}^{d-1,1}\times LG(W=F)$ to the first line of (4.4) gives the un-normalized on-shell three point coupling. To get the normalized on-shell matrix element ${}^{\rm out}\langle 3,2|1\rangle^{\rm in}$ one has to divide it by $\prod_{i=1}^{3}\frac{4m_i}{k}R^{1/2}_{n_i,m_i}$

(see (3.10), (3.11) and (3.13)),

$$^{\rm out}\langle 3,2|1\rangle^{\rm in} = \langle W_{\Delta_3,\bar\Delta_3} e^{ik_3\cdot x} W_{\Delta_2,\bar\Delta_2} e^{ik_2\cdot x} G_{-1/2}\bar G_{-1/2} W_{\Delta_1,\bar\Delta_1} e^{ik_1\cdot x}\rangle \frac{R_{3,2,1}}{\prod_{i=1}^{3}\frac{2(2j_i+1)}{k} R^{1/2}_{n_i,m_i}}\,. \tag{4.21}$$

To recapitulate, we computed the three point coupling of the on-shell states described in section 3 for the case where the momentum around the cigar is zero, $m_i = \bar m_i$, and the winding number is conserved, $\sum m_i = 0$. An incoming state corresponding to (4.15) can decay into two states (4.16) if and only if

$$j_1 \geq j_2 + j_3 + 1\,. \tag{4.22}$$

This selection rule is in fact a consequence of the underlying SL(2) symmetry, i.e. the fact that for principal discrete series representations,

$$|j_2\rangle \otimes |j_3\rangle = \sum_{j_1\geq j_2+j_3+1} |j_1\rangle\,. \tag{4.23}$$

An interesting question is whether the matrix element (4.21) has singularities for different observables. The only possible source of such singularities is $R_{3,2,1}$ (4.19). The first factor in $R_{3,2,1}$, $\Gamma(-i/k)$, is never singular since $i \leq N$ and by using the definition of N, (4.18), and the bounds (2.6) on the j_i one can show that $N < (k-1)/2 < k$. The second factor in (4.19) is likewise regular since $j_1 > N$ (2.6), (4.18), so $2j_1+1-i > 0$ for $i \leq N$. The two remaining Γ functions in the numerator of (4.19) can similarly be shown to be finite for values of j_1, j_2, j_3 which satisfy the unitarity bound (2.6). Thus, the decay amplitude (4.21) is always finite for physical states.

5. Discussion

The d-dimensional Double Scaled Little String Theory defined in [1] has the following properties:

1. A Hagedorn density of states in a theory without gravity.

2. The spectrum of the theory (3.12) includes some massless states, Kaluza-Klein type excitations with masses of order $M_s/\sqrt{k}$, and a stringy spectrum which starts at masses of order M_s.

3. The theory has well defined off-shell Green's functions. The analytic structure of these correlation functions is very reminiscent of local QFT. In particular, the two point functions have poles corresponding to the states (3.12). Moreover, LSZ reduction applies to the three point functions and can be used to study on-shell S-matrix elements.

4. The theory is weakly coupled at moderate energies. The weak coupling approximation breaks down both at high energies and in some cases at very low ones.

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The last point is related to a potentially puzzling aspect of our analysis. Both here and in [1] it was observed that to obtain sensible unitary amplitudes from the weak coupling analysis of DSLST one needs to restrict the observables in external legs of different correlation functions. Examples include the bound (2.6) which is necessary for unitarity of the two point functions, and (4.11) that is needed to make sense of three point functions.[10] These bounds cut off the allowed values of off-shell momenta in the euclidean regime. For example, the stress-tensor of DSLST is described in the bulk theory as an on-shell graviton which is only defined for $k_\mu^2 < 0$.

The above bounds on momenta are clearly inconsistent with analyticity of off-shell Green's functions (it should be emphasized that on-shell S-matrix elements of the states (3.12) *are* analytic in the Mandelstam invariants). LST is a non-local theory, and it is not clear apriori whether it should satisfy off-shell analyticity (see [14] for a recent discussion), but assuming that it does (as is perhaps suggested by property 3 above), the bounds must be reinterpreted. We would like to propose that these bounds are associated with the breakdown of the weak coupling expansion in DSLST. String loops are naively suppressed by powers of $1/\chi$ (1.6), but it is possible that as one approaches the boundaries of the region (2.6), say, loop corrections become large and invalidate the perturbative analysis. Thus, at least some of the bounds (2.6), (4.11) might not be bounds on observables in correlation functions but rather on the reliability of the perturbative analysis of various correlation functions. It would clearly be interesting to compute loop corrections to the two and three point functions we considered here and check this directly.

For the stress-tensor of DSLST described above, string perturbation theory breaks down at long distances ($k_\mu^2 = 0$). It is perhaps not surprising that the bulk description breaks down in this regime, since the "boundary theory" (low energy DSLST) is free at low energies — e.g. in the six dimensional example it is the free theory on k separated NS5-branes. In the AdS/CFT correspondence when the boundary theory is free in the IR, the bulk theory typically becomes strongly coupled there. Usually, the strong coupling is associated with large curvatures in the bulk geometry rather than large corrections from string loops, but we have to recall that the near-horizon geometry of k NS5-branes, which is a multi-center CHS solution [21], is related to the geometry (2.8) by T-duality, which smooths out the large curvature of the multi-center solution, apparently at the price of introducing large string loop corrections.

Unlike anti-de-Sitter space, string theory in asymptotically linear dilaton backgrounds of the sort studied in [9] and here has observables corresponding to wave-functions that are δ function normalizable in the radial (ϕ) direction. These observables are needed to construct off-shell operators in LST with arbitrarily large

[10] In fact, as mentioned above, these bounds are needed in string theory on AdS_3 as well, for similar reasons.

negative d-dimensional momentum k_μ^2. The role of these observables in the theory is puzzling, since they seem to correspond to fluctuating fields in LST and not to external sources, whose norm should diverge from large ϕ. Excluding them seems to put a lower bound on the allowed values of k_μ^2 for each observable, which violates off-shell analyticity. Note that:

1. The issue of the physical nature of the δ function normalizable operators is separate from the discussion of the perturbative unitarity bounds above. In particular, it should not involve strong coupling effects in the "bulk" string theory. Computing the tree level four point function of the observables studied here should shed light on the role of the δ function normalizable operators in the theory.

2. The fate of the δ function normalizable observables is related to another open problem. In [1] and here we discussed the physics of states with a discrete spectrum of masses in d dimensions. The bulk string theory has in addition a continuum of states, which show up in the off-shell correlation functions as branch cuts at values of k_μ^2 corresponding to the δ function normalizable states. The fate of this continuum in LST is hence tied to these observables.

Another set of puzzles is related to the maximally supersymmetric six dimensional case. Consider e.g. k NS5-branes in IIB string theory. We studied the system in a regime where the fivebranes are separated such that the energy of the ground states of D-strings that stretch between the fivebranes, which is of order M_W, is much larger than the other scales in the problem, M_s, $M_s/\sqrt{k}$. One might expect that the fivebranes would not interact in this limit, and the full theory would split into a set of k decoupled theories living on the different fivebranes. It is believed that there is no non-trivial theory on a single fivebrane; therefore, one would expect LST to be trivial for $M_W \gg M_s$. We nevertheless find a non-trivial theory with a complicated k dependent spectrum of masses (3.12) and interactions (the simplest of which — the three point couplings — are described in section 4).

An example of this problem was mentioned at the end of section 3. The analysis of the two point functions in section 3 leads one to conclude that the operators $\mathrm{Tr}\, A^i$ ([17]) create massless single particle states, which therefore carry charge $i = 2, 3, \ldots, k$ under the unbroken SO(2) symmetry of the vacuum. At low energies one expects to find a $\mathrm{U}(1)^{k-1}$ gauge theory with 16 supercharges in which A is a diagonal matrix of scalar fields, whose eigenvalues create single particle states with SO(2) charge 1. Thus, the spectrum of massless particles we find seems to disagree (as far as the SO(2) quantum number is concerned) with what one expects from gauge theory.

One possible resolution of the above discrepancies is that there is a subtlety that has been ignored in the relation of string theory on the manifold (2.8) and the theory on separated fivebranes. For example, it could be that in the fivebrane theory A has an expectation value as well as B, and the SO(2) invariance is restored by summing in some way over different expectation values.

Another possibility is that the identification of the fivebrane system is correct, and our results point to interesting strong coupling effects that exist in this theory at the string scale M_s, even when M_W is very large. This is needed to explain the R-charge spectrum of the massless particles and might also help explain the spectrum (3.12). Imagine turning on M_W continuously in the theory of k initially coincident fivebranes. For $M_W \ll M_s/\sqrt{k}$, the naive gauge theory picture is clearly correct at low energies, and the low lying spectrum includes $k-1$ scalar fields with SO(2) charge 1. As M_W increases and eventually passes M_s, the theory develops a strong coupling regime at energies $M_s/\sqrt{k} < E < M_W$. Thus, it is possible that some operators which are irrelevant at weak coupling become relevant in this energy regime,[11] and cause the theory to flow to a different fixed point in which the number of massless degrees of freedom is the same (due to supersymmetry) but they have different quantum numbers. Examples of such behavior are known to occur in supersymmetric gauge theories (see e.g. [27]).

Note added. The double scaling limit considered in [1] and in this paper is related but different from the one that was studied in the literature on "geometric engineering" (see e.g. [7, 28]), where one typically sends $M_s \to \infty$ keeping the analog of M_W (more precisely χM_s where χ is defined in (1.6)) fixed. In particular, since $M_W \ll M_s$ in this limit, our description of the dynamics is strongly coupled and breaks down beyond the extreme low energy limit. One can of course still study low energy physics as well as properties of BPS states which correspond to D-branes in the geometry (1.8). These aspects of the dynamics have indeed been analyzed in some cases using geometric engineering. The main point of [1] and this paper is that in the opposite limit of $M_W \gg M_s$ the theory is weakly coupled at energy scales well below M_W and one can study non-BPS aspects of the physics like the spectrum and interactions of states at those scales.

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[11] Such operators are known as "dangerously irrelevant" in the general theory of the Renormalization Group.

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A. Some useful formulae

$$\begin{aligned}\Gamma(a+1) &= a\Gamma(a)\,,\\ \Gamma(-n+\epsilon) &= \frac{(-)^n}{\epsilon n!} + O(1)\,, \qquad n = 0,1,2,\dots\end{aligned} \tag{A.1}$$

$$\int d^2x|x|^{2a}x^n|1-x|^{2b}(1-x)^m = \pi\frac{\Gamma(a+n+1)\Gamma(b+m+1)\Gamma(-a-b-1)}{\Gamma(-a)\Gamma(-b)\Gamma(a+b+m+n+2)}\,, \quad n,m\in\mathbb{Z} \tag{A.2}$$

$$\begin{aligned}G(j) &= G(-j-1-k)\,,\\ G(j-1) &= \frac{\Gamma(1+\frac{j}{k})}{\Gamma(-\frac{j}{k})}G(j)\,,\\ G(j-k) &= k^{-(2j+1)}\frac{\Gamma(1+j)}{\Gamma(-j)}G(j)\,.\end{aligned} \tag{A.3}$$

B. On-shell three point couplings

For $N \geq 0$ (4.18), the factor $G(j_1-j_2-j_3-1)$ in (4.8) has a pole (see (4.8), (4.18)). The integral (4.14) supplies the other two poles in (4.17). It can be evaluated by noting that the integrand of (4.14) is a polynomial in $|1-x_2|$. One expands

$$|1-x_2|^{2N} = \sum_{n,m=0}^{N}\binom{N}{n}\binom{N}{m}(-)^{n+m}x_2^n\bar{x}_2^m \tag{B.1}$$

and uses (A.2) to evaluate (4.14). Requiring that this integral has poles at the locations (4.16) constrains n,m in (B.1) to lie in the interval $N+1-n_2 \leq n,m \leq n_3-1$ (and of course (B.1) $0 \leq n,m \leq N$). After some algebra, this leads to (4.20). The other factors in (4.19) arise from (4.8) by substituting (4.15), (4.16) and using (4.8).

To show that the on-shell matrix elements vanish for $N < 0$, one first notes that the function $D(j_3,j_2,j_1;k)$ (4.8) contributes no poles in this case. The factor $G(N)$ in (4.8) is non-singular, since for negative arguments $G(j)$ does not have poles for $j > -k-1$ and one can show that in the regime (2.6), $N > -k-1/2$. The factor $G(-j_1-j_2-j_3-2)$ can become singular for $j_1+j_2+j_3 \geq k-1$, however, as explained in section 4 (see (4.11)) this regime is excluded by unitarity, or alternatively applicability of weakly coupled string theory. The remaining two factors in the numerator of (4.8) have some poles, but those correspond to processes like $2 \leftrightarrow 1+3$ or $3 \leftrightarrow 1+2$.

Thus, to get a non-zero matrix element for $N < 0$ one needs to generate all three poles (4.17) from the integrals (4.14). An explicit calculation shows that one can get at most two.

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